When Almost Is Not Even Close: Remarks on the Approximability of HDTP

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Outline

Parameterized Complexity Theory (1)

Parameterized Decision Problem

An instance of a parameterized decision problem $\mathcal{P}$ is a tuple (x, k), where $x \in \{0, 1\}^*$ is a string describing the problem and $k \in \mathbb{Z}$, which is called the parameter of the problem (codifying other aspects of the problem besides n).

Fixed Parameter Tractability (FPT)

A parameterized decision problem $\mathcal{P}$ is fixed parameter tractable, written $\mathcal{P} \in$ FPT, if it is solvable in time bounded by $f(k) \cdot |x|^{O(1)}$, where $f(k)$ is some computable function of the parameters and $|x|^{O(1)}$ denotes a polynomial of the length of the input.

As suggested by notation, let FPT denote the class of all fixed parameter tractable problems.

Parameterized Complexity Theory (2)

W[1]-membership & W[1]-hardness

W[1] is the class of problems solvable by constant depth combinatorial circuits with at most 1 gate with unbounded fan-in on any path from an input gate to an output gate.
A parameterized problem $\mathcal{P}$ is W[1]-hard if every problem in $W[1]$ is reducible to $\mathcal{P}$ under a parameterized reduction.

W[1] $\neq$ FPT **can be seen as analogous to** P $\neq$ NP **in classical complexity.**[1]

[1] FPT $=$ W[0] and W[i] $\subseteq$ W[j] for all $i \leq j$ (most likely: W[i] $\subset$ W[j]).

Approximation Theory

Approximability Classes

During the course of this presentation, let...

- ...PTAS denote the class of all NP optimization problems that admit a polynomial-time approximation scheme.
- ...APX be the class of NP optimization problems allowing for constant-factor approximation algorithms.
- ...APX-*poly* be the class of NP-optimization problems allowing for polynomial-factor approximation algorithms.

Please note that PTAS $\subseteq$ APX $\subseteq$ APX-*poly* (with each inclusion being proper in case P $\neq$ NP).

Outline

1. ~~Parameterized Complexity & Approximation Theory~~ ✓
2. Heuristic-Driven Theory Projection (HDTP)
3. Complex Analogies: A Parameterized Analysis of HDTP
4. Almost Right: An Approximability Analysis of HDTP
5. Conclusion and Future Work

Heuristic-Driven Theory Projection (1)

Heuristic-Driven Theory Projection (HDTP)

- Computing **analogical relations and inferences** (domains given as many-sorted first-order logic representation/many-sorted term algebras) using a generalization-based approach.
- Base and target of analogy defined in terms of **axiomatisations**, i.e., given by a finite set of formulae.
- Aligning pairs of formulae by means of **anti-unification** (extending classical Plotkin-style first-order anti-unification to a restricted form of higher-order anti-unification).
- Proof-of-concept applications in modeling mathematical reasoning and concept blending in mathematics.

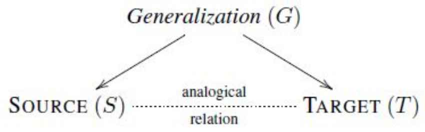

Figure : Analogy-making in HDTP.

Heuristic-Driven Theory Projection (3)

Anti-Unification

- Dual to the unification problem (see, e.g., logic programming or automated theorem proving).
- **Generalizing terms** in a meaningful way, yielding for each term an **anti-instance** (distinct subterms replaced by variables).
- Goal: Finding the **most specific anti-unifier**.
- Plotkin: For a proper definition of generalization, for a given pair of terms there always is exactly one least general generalization (up to renaming of variables).
- Problem: Structural commonalities embedded in different contexts possibly not accessible by first-order anti-unification.

Heuristic-Driven Theory Projection (4)

Restricted Higher-Order Anti-Unification

- First-order terms extended by **introducing variables taking arguments** (first-order variables become variables with arity 0), making a term either a first-order or a higher-order term.
- Class of **substitutions restricted** to (compositions of) the following four cases:

1. Renamings ρ^{F,F^*}: $F(t_1,\dots,t_n) \xrightarrow{\rho^{F,F^*}} F^*(t_1,\dots,t_n)$.
2. Fixations ϕ^F_c: $F(t_1,\dots,t_n) \xrightarrow{\phi^F_f} f(t_1,\dots,t_n)$.
3. Argument insertions $\iota^{F,F^*}_{G,i}$:
$$F(t_1,\dots,t_n) \xrightarrow{\iota^{F,F^*}_{G,i}} F^*(t_1,\dots,t_i,G(t_{i+1},\dots,t_{i+k}),t_{i+k+1},\dots,t_n).$$
4. Permutations π^{F,F^*}_α: $F(t_1,\dots,t_n) \xrightarrow{\pi^{F,F^*}_\alpha} F^*(t_{\alpha(1)},\dots,t_{\alpha(n)})$.

Heuristic-Driven Theory Projection (5)

Examples of higher-order anti-unifications:

$$\begin{tikzcd}[column sep=tiny] & f(X) \arrow[dl, "\rho^X_Y"'] \arrow[dr, "\rho^X_Z"] & \\ f(Y) & & f(Z) \end{tikzcd}$$

(a)

$$\begin{tikzcd}[column sep=tiny] & f(X) \arrow[dl, "\phi^X_a"'] \arrow[dr, "\phi^X_b"] & \\ f(a) & & f(b) \end{tikzcd}$$

(b)

$$\begin{tikzcd}[column sep=tiny] & F(a) \arrow[dl, "\phi^F_f"'] \arrow[dr, "\phi^F_g"] & \\ f(a) & & g(a) \end{tikzcd}$$

(c)

$$\begin{tikzcd}[column sep=tiny] & F(a,b,c) \arrow[dl, "\iota^{F,F'}_{X,2}"'] \arrow[dr, "\iota^{F,F''}_{G,1}"] & \\ F'(a,b,X,c) & & F''(a,G(b,c)) \end{tikzcd}$$

(d)

$$\begin{tikzcd}[column sep=tiny] & F(a,b) \arrow[dl] \arrow[dr, "\pi^{F,F'}_{\alpha}"] & \\ F(a,b) & & F'(b,a) \end{tikzcd}$$

(e)

Heuristic-Driven Theory Projection (6)

Solar System	Rutherford Atom
sorts $real,\ object,\ time$ **entities** $sun : object,\ planet : object$ **functions** $mass : object \to real \times \{kg\}$ $dist : object \times object \times time \to real \times \{m\}$ $force : object \times object \times time \to real \times \{N\}$ $gravity : object \times object \times time \to real \times \{N\}$ $centrifugal : object \times object \times time \to real \times \{N\}$ **predicates** $revolves\_around : object \times object$ **facts** $\alpha_1 : mass(sun) > mass(planet)$ $\alpha_2 : mass(planet) > 0$ $\alpha_3 : \forall t : time : gravity(planet, sun, t) > 0$ $\alpha_4 : \forall t : time : dist(planet, sun, t) > 0$ **laws** $\alpha_5 : \forall t : time,\ o_1 : object,\ o_2 : object :$ $dist(o_1, o_2, t) > 0 \wedge gravity(o_1, o_2, t) > 0$ $\to centrifugal(o_1, o_2, t) = -gravity(o_1, o_2, t)$ $\alpha_6 : \forall t : time,\ o_1 : object,\ o_2 : object :$ $0 < mass(o_1) < mass(o_2) \wedge dist(o_1, o_2, t) > 0 \wedge$ $centrifugal(o_1, o_2, t) < 0$ $\to revolves\_around(o_1, o_2)$	**sorts** $real,\ object,\ time$ **entities** $nucleus : object,\ electron : object$ **functions** $mass : object \to real \times \{kg\}$ $dist : object \times object \times time \to real \times \{m\}$ $coulomb : object \times object \times time \to real \times \{N\}$ **facts** $\beta_1 : mass(nucleus) > mass(electron)$ $\beta_2 : mass(electron) > 0$ $\beta_3 : \forall t : time : coulomb(electron, nucleus, t) > 0$ $\beta_4 : \forall t : time : dist(electron, nucleus, t) > 0$

Heuristic-Driven Theory Projection (7)

types
$real,\ object,\ time$
constants
$X : object,\ Y : object$
functions
$mass : object \longrightarrow real \times \{kg\}$
$dist : object \times object \times time \rightarrow real \times \{m\}$
$F : object \times object \times time \rightarrow real \times \{N\}$
$centrifugal : object \times object \times time \longrightarrow real \times \{N\}$
predicates
$revolves\_around : object \times object \times object$
facts
$\gamma_1 : mass(X) > mass(Y)$
$\gamma_2 : mass(Y) > 0$
$\gamma_3 : \forall t : time : F(X, Y, t) > 0$
$\gamma_4 : \forall t : time : dist(X, Y, t) > 0$
laws
$\gamma_5* : \forall t : time,\ o_1 : object,\ o_2 : object :$
$dist(o_1, o_2, t) > 0 \wedge F(o_1, o_2, t) > 0$
$\rightarrow centrifugal(o_1, o_2, t) = -F(o_1, o_2, t)$
$\gamma_6* : \forall t : time,\ o_1 : object,\ o_2 : object :$
$0 < mass(o_1) < mass(o_2) \wedge dist(o_1, o_2, t) > 0 \wedge\ centrifugal(o_1, o_2, t) < 0$
$\rightarrow revolves\_around(o_1, o_2)$

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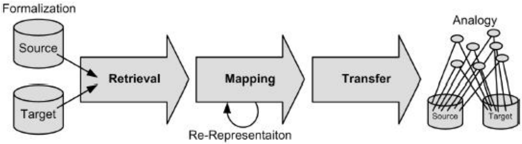
Complexity of HDTP (1)

HDTP is naturally split into two mechanisms:

- Analogical matching of input theories.
- Re-representation of input theories by deduction in FOL.



⇒ Re-representation is undecidable (undecidability of FOL).

⇒ Focus on mechanism for analogical matching.

Complexity of HDTP (2)

Problem 1. F Anti-Unification
Input: Two terms f, g, and a natural $k \in \mathbb{N}$
Problem: Is there an anti-unifier h, containing at least k variables, using only **renamings and fixations**?

Problem 2. FP Anti-Unification
Input: Two terms f, g, and naturals $l, m, p \in \mathbb{N}$.
Problem: Is there an anti-unifier h, containing at least l 0-ary variables and at least m higher arity variables, and two substitutions σ, τ using only renamings, fixations, and **at most** p **permutations** such that $h \xrightarrow{\sigma} f$ and $h \xrightarrow{\tau} g$?

Problem 3. FPA Anti-Unification
Input: Two terms f, g and naturals $l, m, p, a \in \mathbb{N}$.
Problem: Is there an anti-unifier h, containing at least l 0-ary variables, at least m higher arity variables, and two substitutions σ, τ using renamings, fixations, at most p permutations, and **at most** a **argument insertions** such that $h \xrightarrow{\sigma} f$ and $h \xrightarrow{\tau} g$?

Complexity of HDTP (3)

Complexity of HDTP (Higher-Order Anti-Unification)

1. **F Anti-Unification is solvable in polynomial time**.
2. Let m be the maximum number of higher arity variables and p be the maximum number of permutations applied. Then **FP Anti-Unification is** NP**-complete and** W[1]**-hard w.r.t. parameter set** $\{m, p\}$.
3. Let r be the maximum arity and s be the maximum number of subterms of the input terms. Then **FP Anti-Unification is in** FPT **w.r.t. parameter set** $\{s, r, p\}$.
4. **FPA Anti-Unification is** NP**-complete and** W[1]**-hard w.r.t. parameter set** $\{m, p, a\}$.

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Approximability Analysis (1)

- FP Anti-Unification W[1]-hard to compute for parameter set m, p (m number of higher-arity variables, p number of permutations).
  $\Rightarrow$ No polynomial-time algorithm computing "sufficiently complex" generalizations (i.e., with lower bound on number of higher-arity variables), upper bounding number of permutations (W[1]-hardness for single permutation).
- **What if one considers generalizations which merely approximate the "optimal" generalization in some sense?**

Approximability Analysis (2)

Complexity of a Substitution

The complexity of a basic substitution σ is defined as

$$C(s) = \begin{cases} 0, & \text{if } \sigma \text{ is a renaming.} \\ 1, & \text{if } \sigma \text{ is a fixation or permutation.} \\ k+1, & \text{if } \sigma \text{ is a } k\text{-ary argument insertion.} \end{cases}$$

The complexity of a restricted substitution $\sigma = \sigma_1 \circ \cdots \circ \sigma_n$ (i.e., the composition of any sequence of unit substitutions) is the sum of the composed substitutions: $C(\sigma) = \sum_{i=1}^{n} C(\sigma_i)$.

Approximability Analysis (3)

Consider problem of finding generalization which **maximizes complexity over all generalizations**:

- Complex generalization would contain "most information" present over all of the generalizations chosen (i.e., maximizing the "information load").
- Using approximability results on MAXCLIQUE:

Approximation Complexity of HDTP Analogy-Making

FP anti-unification is not in APX **(i.e., is hard for** APX**-poly).**

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Conclusion and Future Work

- Analogy-making in HDTP is widely not tractable or approximable!
- Taking the Tractable AGI Thesis (**see talk tomorrow afternoon!**) into account, the suitability of HDTP as basis for a general model for high-level cognitive capacities or a general cognitive architecture seems questionable.
- Main question(s) for future research:
  - How can the computation of generalizations via restricted higher-order anti-unification be constrained in a meaningful way as to remain polynomially-solvable?
  - Similarly: How can the underlying KR formalism be restrained/modified?
  - How can the undecidability of the re-representation mechanism directly be addressed (and mitigated)?

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